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Study Material (Unit 1: Matrices)

Updated on 18/09/2026

Syllabus: Unit 1: Matrices: Inverse and Rank of matrices using elementary transformations, system of simultaneous linear equations, Symmetric, Skew-symmetric and Orthogonal Matrices; Hermitian and Skew Hermitian Matrices, Characteristic equation, Cayley-Hamilton theorem and its applications, Eigen values and Eigenvectors; and Diagonalization of a Matrix.

Inverse and Rank of matrices using elementary transformations

Lecture 1: Elementary Transformations and Inverse of a matrix

Any one of the following operations on a matrix is called an elementary transformation

1. Interchanging any two rows (columns). indicated by Ri,j if i^{th} & j^{th} rows are interchanged.
2. Multiplication of the elements of any row or column by a non-zero scalar quantity k (kRj)
3. Addition of constant multiplication of the elements of any row to the corresponding elements of any other row. (Ri + kRj).

Two metrics are said to be equal if one is obtained from the other by elementary transformations.

Elementary Matrix: A matrix obtained from a unit matrix by elementary transformation is called elementary matrix.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $R_2 + 2R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is called the elementary matrix.

To find inverse of a matrix employing elementary transformations:

If A is reduced to I by elementary transformation, then

PA = I

Working Rule: Write A = IA. Apply elementary row transformations on A of the left side and on I of the right hand side so that A is reduced to I and I of R.H.S. reduced to P getting I = PA, then P is inverse of A.

Problem : By elementary row transformation find the inverse of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Now $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

or $R_3 \to R_3 - 3R_1$

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix}$ A

or $R_3 \to \frac{1}{2} R_3$

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$ A

or $R_2 \to R_2 - 2R_3, R_1 \to R_1 - 3R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{15}{2} & \frac{11}{2} & -\frac{3}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -3/2 & 1/2 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -3/2 & 1/2 \end{bmatrix} A$$

I = PA

So $P = A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -3/2 & 1/2 \end{bmatrix}$

Lecture 2: Rank of matrices using elementary transformations

Let A be m x n matrix. If A is a null matrix, we define its rank to be '0'. If A is a non-zero matrix, we say that 'r' is the rank of A if

i. Every (r + 1) th order minor of A is '0' (zero) &
ii. At least one rth order minor of A which is not zero.

It is denoted by ρ (A) and read as rank of A.

Example: Find the rank of the matrix

$$\begin{pmatrix} 1 & 5 \\ 3 & 9 \end{pmatrix}$$

Solution: Let A=

$$\begin{pmatrix} 1 & 5 \\ 3 & 9 \end{pmatrix}$$

Order of A is $2 \times 2 \therefore \quad \rho(A) \leq 2$

Consider the second order minor

$$\begin{vmatrix} 1 & 5 \\ 3 & 9 \end{vmatrix} = -6 \neq 0$$

There is a minor of order 2, which is not zero. $\therefore \rho(A) = 2$

Example: Find the rank of the matrix

$$\begin{pmatrix} -5 & -7 \\ 5 & 7 \end{pmatrix}$$

Solution: Let A=

$$\begin{pmatrix} -5 & -7 \\ 5 & 7 \end{pmatrix}$$

Order of A is $2 \times 2 \therefore \rho(A) \leq 2$

Consider the second order minor

$$\begin{vmatrix} -5 & -7 \\ 5 & 7 \end{vmatrix} = 0$$

Since the second order minor vanishes, $\rho(A) \neq 2$

Consider a first order minor $|-5| \neq 0$

There is a minor of order 1, which is not zero

$\therefore \rho(A) = 1$

Example: Find the rank of the matrix

$$\begin{pmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{pmatrix}$$

Solution: Let A=

$$\begin{pmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{pmatrix}$$

Order of A is 3x3

$\therefore \rho(A) \leq 3$

Consider the third order minor $\begin{vmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{vmatrix}$

$= 6 \neq 0$

There is a minor of order 3, which is not zero

$\therefore \rho(A) = 3.$

Example: Find the rank of $\begin{pmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{pmatrix}$

Solution: Let A=

$$\begin{pmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{pmatrix}$$

Order of A is 3x3

$\therefore \rho(A) \leq 3$

Consider the third order minor

$$\begin{vmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{vmatrix} = 0$$

Since the third order minor vanishes, therefore $\rho(A) \neq 3$
Consider a second order minor

$$\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 7 \neq 0$$

There is a minor of order 2, which is not zero.

$\therefore \rho(A) = 2$.

Example: Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$$

Solution: Let A =

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$$

Order of A is 3×4

$\therefore \rho(A) \leq 3$.

Consider the third order minors

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ 3 & 6 & 3 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \\ 3 & 3 & -7 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & -2 \\ 3 & 6 & -7 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ 6 & 3 & -7 \end{vmatrix} = 0$$

Since all third order minors vanishes, $\rho(A) \neq 3$.

Now, let us consider the second order minors,

$$\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 6 \neq 0$$

There is a minor of order 2 which is not zero.

$\therefore \rho(A) = 2$.

Echelon form: A matrix is said to be in Echelon form, if

1. Zero rows, if any exists, they should be below the non-zero row.
2. The first non-zero entry in each non-zero row is equal to '1'.
3. The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Note: The number of non-zero rows in echelon form of A is the rank of 'A'.

Example: Find the rank of the matrix A=

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$$

Solution:

The order of A is 3×3.

$\therefore \rho(A) \leq 3$.

Let us transform the matrix A to an echelon form by using elementary transformations.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - R_2$
The above matrix is in echelon form	

The number of non-zero rows is 2

∴Rank of *A is* 2.

ρ (*A*) = 2.

Example: Find the rank of the matrix A=

$$\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$$

Solution:

The order of A is 3×4.

$\therefore \rho(A) \leq 3$.

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	
$A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	$R_1 \leftrightarrow R_2$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{pmatrix}$	$R_3 \rightarrow R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$	$R_3 \rightarrow R_3 + 5R_2$

The number of non-zero rows is 3. $\therefore \rho(A) = 3$.

Example: Find the rank of the matrix A=

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{pmatrix}$$

Solution:

The order of A is 3×4.

$\therefore \rho(A) \leq 3.$

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -2 \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$R_3 \to R_3 - R_2$

The number of non-zero rows is 3.

$\therefore \rho(A) = 3.$

Normal form: Every non-zero Matrix can be reduced to any one of the following forms

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I_r \\ 0 \end{bmatrix}, \begin{bmatrix} I_r & 0 \end{bmatrix}, \begin{bmatrix} I_r \end{bmatrix}$$

known as normal forms or canonical forms by using Elementary transformations where I_r is the unit matrix of order 'r' and 'O' is the null matrix.

Note: In this form "the rank of a matrix is equal to the order of an identity matrix.

Example: Find the rank of the matrix $A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$, by reducing it to normal form.

Solution. We have $A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix}$ $\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{matrix}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix} \begin{matrix} C_2 \to C_2 - 3C_1 \\ C_3 \to C_3 - 4C_1 \\ C_4 \to C_4 - 2C_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 3R_2 \end{matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \leftrightarrow R_4$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} C_3 \to C_3 - \frac{5}{7}C_2 \\ C_4 \to C_4 - \frac{2}{7}C_2 \end{matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to -\frac{1}{7}R_2 \\ R_3 \to -R_3 \end{matrix}$$

$= \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ which is normal form.

$\therefore \rho(A) = 3$.

Lecture 3: Consistency of linear system of equations and their solution:

Linear equations are

Consistent: A system of equations is said to be consistent,

if they have one or more solutions.

x + 2y = 4	x + 2y = 4
3x + 2y = 2	4x + 8y = 16
Unique solution	Infinite Solution

Inconsistent: If a system of equation has no solution then it is said to be inconsistent

x + 2y = 4

3x + 6y = 5

Consider the system of Linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots\ldots\ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots\ldots\ldots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots\ldots\ldots\ldots + a_{3n}x_n = b_3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots\ldots\ldots\ldots\ldots + a_{mn}x_n = b_m$$

$$\text{or} \begin{bmatrix} a_{11} & a_{12} & .....a_{1n} \\ a_{21} & a_{22} & .....a_{2n} \\ ..... & ..... & ..... \\ a_{m1} & a_{m2} & .....a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ .... \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ .... \\ b_m \end{bmatrix}$$

or AX = B

$$\text{and } C = [A:B] = \begin{bmatrix} a_{11} & a_{12} & .....a_{1n} : b_1 \\ a_{21} & a_{22} & .....a_{2n} : b_2 \\ ..... & ..... & ..... \\ a_{m1} & a_{m2} & .....a_{mn} : b_m \end{bmatrix}$$

is called the augmented matrix

(a) Consistent Equation: If rank A = Rank C

(1) Unique solution if Rank A = Rank C = n

(if number of unknowns)

(2) Infinite solution if Rank A = Rank C = r, r<n

(b) Inconsistent equation: If Rank A ≠ Rank C

Problem: Show that the equations 2x + 6y = -11, 6x + 20y - 6z = -3, 6y - 18z = -1 are not consistent.

Solution: Augmented Matrix C = [A: B]

$$= \begin{bmatrix} 2 & 6 & 0:-11 \\ 6 & 20 & -6:-3 \\ 0 & 6 & -18:-1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 0:-11 \\ 0 & 2 & -6:30 \\ 0 & 6 & -18:-1 \end{bmatrix} \quad R_2 \to R_2 - 3R_1$$

$$R_3 \to R_3 - 3R_2 \begin{bmatrix} 2 & 6 & 0:-11 \\ 0 & 2 & -6:30 \\ 0 & 6 & 0:-91 \end{bmatrix}$$

The rank of C is 3 and rank of A is 2

Hence Rank A ≠ Rank C so equations are not consistent.

Problem: Determine for what values of and the following equations have (1) Non Solution (2) A unique solution (3) Infinite number of solutions.

x + y + z = 6, x + 2y + 3z = 10, x + 2y + λ z = μ

Solution : $\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ 10\\ \mu\end{bmatrix}$

$$AX = B$$

$$C=[A:B]=\begin{bmatrix}1 & 1 & 1:6\\ 1 & 2 & 3:10\\ 1 & 2 & \lambda:\mu\end{bmatrix}\frac{R_2\to R_2-R_1}{R_3\to R_3-R_1}\to\begin{bmatrix}1 & 1 & 1:6\\ 0 & 1 & 2:6\\ 0 & 1 & \lambda-1:\mu-6\end{bmatrix}$$

$$\xrightarrow{R_3\to R_3-R_1}\begin{bmatrix}1 & 1 & 1:6\\ 0 & 1 & 2:4\\ 0 & 0 & \lambda-3:\mu-10\end{bmatrix}$$

(1) There is no solution if R (A) ≠ R (C)

i.e. $\lambda-3=0$ *or* $\lambda=3$ *and*

$\mu-10\neq 0$ *or* $\mu\neq 10$

(2) There is unique solution if R (A) = R (C) =3

i.e. $\lambda-3\neq 0$ *or* $\lambda\neq 3$, μ may have any value.

(3) Infinite number of Solution is R (A) = R (C) = 2

$\lambda=3,\ \mu=10$.

Linear dependents and independence of vectors:

Vectors (Matrix) X_1, X_2 .....X_n are said to be dependent if

(1) all the vectors (row or column matrices) are of same order.

(2) n scalars $\lambda_1,\lambda_2.....\lambda_n$ (nor all zero) exist such that

$$\lambda_1X_1+\lambda_2X_2+\lambda_3X_3+.....\lambda_nX_n=0$$

otherwise they are linearly independent.

Problem: Check for linear dependence of system of vectors.

$$X_1=\begin{bmatrix}2\\ 2\\ 1\end{bmatrix}\quad X_2=\begin{bmatrix}1\\ 3\\ 1\end{bmatrix}\quad X_3=\begin{bmatrix}1\\ 2\\ 2\end{bmatrix}$$

Solutions : Consider the matrix equation $\lambda_1X_1+\lambda_2X_2+\lambda_3X_3=0$

$$\lambda_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad \lambda_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad \lambda_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2\lambda_1 + \lambda_2 + \lambda_3 = 0$$
$$2\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0$$
$$\lambda_1 + \lambda_2 + 2\lambda_3 = 0 \;\; a \;\; \text{hom}\, ogeneous \;\; equation$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow[or]{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 + \lambda_2 + 2\lambda_3 = 0$$
$$\lambda_2 + 2\lambda_3 = 0$$
$$-5\lambda_3 = 0 \Rightarrow \lambda_3 = 0$$
$$\Rightarrow \lambda_2 = 0$$
$$\Rightarrow \lambda_1 = 0$$

so any one of $\lambda_1, \lambda_2, \lambda_3$ are not different from zero.

So system of equations not linearly independent.

Homogeneous Equation: Let $A_{m\times n}$ be the coefficient matrix.

Homogeneous equation.

$$\boxed{AX = 0}$$

$$\boxed{m = n,\ m > 3,\ n > 3} \qquad \boxed{m = n = 3}$$

Rank A = n Trivial Solution Unique Solution	Rank A < n Infinite Solution	$\lvert A \rvert \neq 0$ Trivial Solution

Pr of su

$$\lambda x + 2y - 2z - 1 = 0,\ 4x + 2\lambda y - z - 2 = 0$$
$$6x + 6y + \lambda z - 3 = 0$$

Solution:

$$\begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The given system of equation has a unique solution if and only if the coefficient matrix is non singular

i.e. $\begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix} \neq 0 \Rightarrow \lambda^3 + 11\lambda - 30 \neq 0$

$$\Rightarrow (\lambda - 2)(\lambda^2 + 2\lambda + 15) \neq 0$$
$$\Rightarrow \lambda \neq 2$$

Therefore if $\lambda \neq 2$, the given system of equations have a unuque solution.

If $\lambda = 2$

$$\begin{bmatrix} 2 & 2 & -2 \\ 4 & 4 & -1 \\ 6 & 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \frac{R_2 \to R_2 - 2R_1}{R_3 \to R_3 - 3R_1} \to \begin{bmatrix} 2 & 2 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{8}{3}R_3 \begin{bmatrix} 2 & 2 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x + 2y - 2z = 1,\ 3z = 0$$

$$\Rightarrow 2x + 2y = 1 \quad \text{if } y = k,\ x = \frac{1}{2} - k$$

Lecture 4: Various Types of matrices

(1) Row Matrix: If a matrix has only one row and any number of columns then it is called a row matrix. For ex. [2, 7, 3, 9]

(2) Column Matrix: A matrix having one column and any number of rows is called a column matrix.

For ex. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(3) Null Matrix or Zero matrix : Any matrix, in which the elements are zeros, is called a zero matrix or null matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(4) Square matrix : A matrix, in which the number of rows is equal to the number of columns, is called square matrix i.e. $\begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix}$

(5) Diagonal Matrix : A square matrix is called a diagonal matrix, if all its non-diagonal elements are zero e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(6) Unit or Identity Matrix : A square matrix is called a unit matrix if all the diagonal elements are unity and non-diagonal elements are zero e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(7) Symmetric Matrix: A square matrix is said to be symmetric if for all values of i and j, aji = aji e.e. A' = A

e.g. $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

(8) Skew Symmetric matrix: A square matrix is said to be Skew Symmetric Matrix

a) if for all values of i & j, aij = - aji or A' = -1

b) All diagonal elements are zero e.g. $\begin{bmatrix} o & -h & -g \\ h & o & -f \\ g & f & o \end{bmatrix}$

(9) Triangular Matrix (Echelon Form): A square matrix all of whose elements below the principal diagonal are zero is called an upper Triangular matrix. A square matrix all of whose elements above the principal diagonal elements are zero, is called a lower triangular matrix. i.e.

Upper Triangular matrix is $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ Lower triangular matrix $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 6 & 1 & 2 \end{bmatrix}$

(10) Transpose of a Matrix: On interchanging the rows and columns, the new matrix obtained is called transpose matrix of the given matrix A and is denoted by A'.

For ex. If $A=\begin{bmatrix}2&0&0\\3&4&0\\6&1&2\end{bmatrix}$ *then* $A'=\begin{bmatrix}2&0&6\\3&0&7\\4&5&8\end{bmatrix}$

(11) Orthogonal Matrix: A square matrix A is called a orthogonal matrix if the product of the matrix A and the transpose matrix A' is an identity matrix i.e.

A.A'=I if $|A|$ then the matrix A is proper.

(12) Conjugate of a Matrix : Let $A=\begin{bmatrix}1+i&2-i\\7+2i&-i\end{bmatrix}$

then conjugate of matrix A is $\bar{A}=\begin{bmatrix}1-i&2+3i\\7-2i&i\end{bmatrix}$

(13) Transposed conjugate of a matrix A^{θ}:

Transpose of the conjugate of a matrix is denoted by A^{θ}

For ex. $A=\begin{bmatrix}1+i&2-3i\\7+2i&-i\end{bmatrix}$ *then* $\bar{A}=\begin{bmatrix}1-i&2+3i\\7-2i&i\end{bmatrix}$

$\left(\bar{A}\right)=A^{\theta}=\begin{bmatrix}1-i&7-2i\\2+3i&i\end{bmatrix}$

(14) Unitary Matrix : A square matrix A is said to be unitary if $A^{\theta}A=I$

For ex. $A=\begin{bmatrix}\frac{1+i}{2}&\frac{-1+i}{2}\\\frac{1+i}{2}&\frac{1-i}{2}\end{bmatrix}$ *then* $A^{\theta}=\begin{bmatrix}\frac{1-i}{2}&\frac{-1-i}{2}\\\frac{-1-i}{2}&\frac{1+i}{2}\end{bmatrix}$ & $AA^{\theta}=I$

(15) Hermitian Matrix: A square matrix A = [aij] is called a Hermitian matrix, if every *i* - j^{th} element of A is equal to conjugate complex *j*-i^{th} element of A.

i.e. $aij=\bar{a}ji$

i.e. $\begin{bmatrix}1&2+3i&3+i\\2-3i&2&1-2i\\3-i&1+2i&5\end{bmatrix}$

Necessary & sufficient condition for a matrix A to be Hermitian is that $A=A^{\theta}$ *i.e.* $A=\left(\bar{A}\right)'$

(16) Spew Hermitian Matrix: A square matrix A = (aij) is called a skew Hermitian matrix if every i-j^{th} element of A is equal to negative of complex conjugate of j - i^{th} element of A. In other words, $aij = -\bar{a}ji$

Remark : All the diagonal elements of a spew Hermitian matrix are either zero or pure imaginary $\begin{bmatrix} i & 2-3i \\ -(2+3i) & 0 \end{bmatrix}$

The necessary and sufficient condition for a matrix A to be spew Hermitian is that $A^{\theta} = -A$ *or* $\left(\bar{A}\right)' = -A$

(17) Idempotent Matrix: A matrix such that $A^2 = A$ is called idempotent matrix

e.g. $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

(18) Periodic Matrix: A matrix A will be called a periodic matrix if $A^{k+1} = A$

If k is least the integer for which $A^{k+1} = A$ then k is said to be period of A.

(19) **Nilpotent Matrix:** A matrix will be called nilpotent if $A^k = 0$ where k is the integer.

If k is the least +ve integer for which $A^k = 0$ is called index of the nilpotent matrix e.g. If $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ *then* $A^2 = 0$

Hence A is a nilpotent matrix whose index is 2.

(20) **Involutary matrix:** A matrix for which $A^2 = I$ (unit matrix), is called involutary matrix.

e.g. Unit matrix is involutary.

(21) **Equal Matrix:** Two matrices are said to be equal if

(1) They are of same order

(2) Elements in the corresponding position are equal.

Thus if $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ *then* $A = B$

Unitary Matrix : A square matrix A is said to be unitary matrix if $AA^{\theta} = A^{\theta}A = I$

Problem : Prove that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is a unitary matrix.

Proof : $A=\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$, $A^{\theta}=\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

So $A^{\theta}A=\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\times\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$=\frac{1}{\sqrt{3}}\begin{bmatrix} 1+(1+1) & (1+i)-(1+i) \\ (1-i)-i(1-i) & (1+1)+1 \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=1$$

Hence A is a unitary matrix.

Problem: The modules of each characteristic root of a unitary matrix is unity.

Proof : Since A is a unitary matrix so $A^{\theta}A=I$

Let λ be characteristic root of A, then $AX=\lambda X$ ..... (1)

Taking conjugate transpose of both sides we get $(AX)^{\theta}=\bar{\lambda}X^{\theta}$

or $X^{\theta}A^{\theta}=\bar{\lambda}X^{\theta}$ ..... (2) From (1) & (2) $(X^{\theta}A^{\theta})(AX)=\bar{\lambda}\lambda X^{\theta}X$

or $X^{\theta}(A^{\theta}A)X=\bar{\lambda}\lambda X^{\theta}X$ *or* $X^{\theta}IX=\bar{\lambda}\lambda X^{\theta}X$ *or* $X^{\theta}X(\bar{\lambda}\lambda-1)=0\Rightarrow\lambda\bar{\lambda}=1$

So $X^{\theta}(A^{\theta}A)X=\bar{\lambda}\lambda X^{\theta}X$ *or* $\lambda^2=1\Rightarrow|\lambda|=1$ *as* $X^{\theta}X\neq 0$

Lecture 5: Eigen Values

1- Characteristic Roots or Eigen Values:

(a) For a given square matrix A, $A-\lambda I$ matrix is called the characteristic matrix, where is a scalar and I is a unit matrix.

Let $A=\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, $A-\lambda I=\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$ is characteristic matrix.

(b) **Characteristic polynomial :** The determinant $|A-\lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix A.

For ex. $\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}=(2-\lambda)(6-5\lambda)+\lambda^2-2)-2(2-\lambda-1)+(2-3+\lambda)$

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5$$

(c) **Characteristic Equation :** The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A. e.g. $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

(d) **Characteristic Roots or Eigen Values :** The roots of the characteristic equation $|A - \lambda I| = 0$ are called characteristic roots of matrix A. e.g. $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$(\lambda - 1)(\lambda - 1)(\lambda - 5) = 0$ *so* $\lambda = 1, 1, 5$

So characteristic roots are 1, 1, 5.

Remark on Eigen Values:

1. Any square matrix A and its transpose A' have the same Eigen value.
2. Sum of Eigen values of a matrix is equal to the trace of the matrix.
   Sum of the elements on the principal diagonal of a matrix is called the trace of the matrix.
3. The product of the Eigen values of a matrix A is equal to the determinant of A.
4. If $\lambda_1, \lambda_2 ..... \lambda_n$ are the Eigen values of A, then the Eigen values of
   (1) kA are $k\lambda_1, k\lambda_2 ..... k\lambda_n$
   (2) A^m are $\lambda_1^m, \lambda_2^m ..... \lambda_n^m$
   (3) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ... \frac{1}{\lambda_n}$

Proof of (1) : Characteristic equation of A is $|A - \lambda I| = 0$ characteristic equation of matrix A' is $|A' - \lambda I| = 0$ clearly both are same so $|A| = |A'|$

Proof (2) & (3) : Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$|A - \lambda I| = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix}$$

$$|A - \lambda I| = \lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) + \lambda(-a_{11}a_{22} - a_{11}a_{33} + a_{12}a_{21} - a_{22}a_{33} + a_{23}a_{32} + a_{33}a_{31})$$

$-\left[a_{11}\left(a_{22}a_{33}-a_{23}a_{32}\right)-a_{12}\left(a_{21}a_{33}-a_{23}a_{31}\right)+a_{13}\left(a_{21}a_{32}-a_{31}a_{22}\right)\right]$

If $\lambda_1,\lambda_2,\lambda_3$ be the roots of the equation (1) then sum of the roots $\lambda_1+\lambda_2+\lambda_3=a_{11}+a_{22}+a_{33}$ sum of diagonal element.

Product of the roots $\left[a_{11}\left(a_{22}a_{33}-a_{23}a_{32}\right)-a_{12}\left(a_{21}a_{33}-a_{23}a_{31}\right)+a_{13}\left(a_{21}a_{32}-a_{31}a_{22}\right)\right]$

= Determinant A

Problem: If A and P be square matrices of same type and if P is invertible, show that the matrices A and $P^{-1}AP$ have the same characteristic root.

Solution: Let us put $B = P^{-1}AP$ then we have to show that A and B have same characteristic root

$$B-\lambda I = p^{-1}AP-\lambda I = p^{-1}AP-p^{-1}(A-\lambda I)P$$

$$|B-\lambda I| = |(A-\lambda I)P| = |p^{-1}||A-\lambda I||P|$$

$$= |(A-\lambda I)||p^{-1}||P|$$

$$= |(A-\lambda I)||p^{-1}p|$$

$$= |(A-\lambda I)||I|$$

$$= |(A-\lambda I)| \quad as \quad |I|=1$$

Thus A & B have same characteristic roots.

Problem: Show that characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

Solution: Consider the triangular matrix

$$A=\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

so $|A-\lambda I|=0$ *or* $\begin{bmatrix} a_{11}-\lambda & 0 & 0 & 0 \\ a_{21} & a_{22}-\lambda & 0 & 0 \\ a_{31} & a_{32} & a_{33}-\lambda & 0 \\ a_{41} & a_{42} & a_{43} & a_{44}-\lambda \end{bmatrix}$

on expansion

or $(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)(a_{44}-\lambda)=0$

so $\lambda = a_{11}, a_{22}, a_{33}, a_{44}$

which are the diagonal elements of matrix A.

Problem: the characteristic roots of a Hermitian matrix are all real.

Solution: Since matrix A is Hermitian so $A^\theta = A$

Also $\left(\lambda A^\theta\right) = \bar{\lambda} A^\theta$ & $(AB)^\theta = B^\theta A^\theta$

If is a characteristic root of matrix A then $AX = \lambda X$

or $X^\theta AX = X\lambda X$

Now taking in (1)

$$\left(AX\right)^\theta = \left(\lambda X\right)^\theta \quad or \quad X^\theta A^\theta = \bar{\lambda} X^\theta$$

as A is Hermitian $A^\theta = A$

so $X^\theta A = \bar{\lambda} X^\theta$ *or* $X^\theta AX = \bar{\lambda} X^\theta X$ ..... (2)

$IX^\theta AX = X^\theta \lambda X = \lambda X^\theta X$ ..... (3)

Hence by (2) & (3) $\bar{\lambda} = \lambda$ showing is real.

Deduction I: characteristic roots of real symmetric matrix are all real

as for symmetric, $A' = A \quad \left(\bar{A}'\right)\bar{A}$

or $A^\theta = A \quad \therefore \quad \left(\bar{A}\right) = A$ as A is real

Deduction II: Characteristic of a skew Hermitian matrix is either Zero or pure imaginary.

Proof: If A is spew Hermitian then iA is Hermitian.

Also λ is a characteristic root of A then $AX = \lambda X$

$$\therefore (iA)X = (i\lambda)X$$

so λ is characteristic root of matrix iA which is Hermitian and hence $i\lambda$ should be real which possible only when λ is pure imaginary or zero.

Problem: Modulus of each characteristic root of a unitary matrix is unity.

Solution : Let A be unitary matrix then $A^\theta A = I$

If λ be characteristic root of A, then $AX = \lambda X$ Taking transposed conjugate on both sides

$$(AX)^\theta = (\lambda X)^\theta \quad or \quad X^\theta A^\theta = \bar{\lambda} X^\theta$$

$$X^\theta A^\theta AX = \bar{\lambda} X^\theta \lambda X = \bar{\lambda}\lambda X^\theta X$$

But $A^\theta A = I$

so $X^{\theta}IX = \bar{\lambda}\lambda X^{\theta}X$ *or* $X^{\theta}X = \bar{\lambda}\lambda X^{\theta}X$

$$\therefore (1-\bar{\lambda}\lambda)X^{\theta}X = 0$$

But $X \neq 0 \therefore X^{\theta}X \neq 0$ hence $1-\lambda\bar{\lambda} = 0$ *or* $1 = \lambda\bar{\lambda}$

or $|\lambda|^2 = 1$

so characteristic root is unimodular.

Lecture 6: Caley-Hamilton Theorem

Every square matrix satisfies it sown characteristic equation.

If $|A-\lambda I| = (-1)^2\left(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_n\right)$ be the characteristic polynomial of n×n matrix

A = (aij), then the matrix equation

$X^n + a_1X^{n-1} + a_2X^{n-2} + ... + a_nI = 0$ is satisfied

by X = A i.e.

$$A^n + a_1A^{n-1} + a_2A^{n-2} + ... + a_nI = 0$$

Proof : Since the elements of $A-\lambda I$ are at most of the first degree in λ, the elements of adj $(A-\lambda I)$ are at most degree (n-1) in λ. Thus adj $(A-\lambda I)$ may be written as a matrix polynomial in λ,

$$adj(A-\lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + ... + B_{n-1}$$

where $B_0, B_1, ...B_{n-1}$ are $n \times n$ matrices, their elements being polynomial in λ.

we know that $(A-\lambda I)\,Adj(A-\lambda I) = |A-\lambda I|\,I$

$$(A-\lambda I)\left(B_0\lambda^{n-1} + B_1\lambda^{n-2} + B_{n-1}\right) = (-1)^n(\lambda^n + a_1\lambda^{n-1} + ... + a_n)I$$

Equating coefficients of like powers of on both sides we get

$$-IB_0 = (-1)^n I$$
$$AB_0 - IB_1 = (-1)^n a_1 I$$
$$AB_1 - IB_2 = (-1)^n a_2 I$$
$$.....$$
$$AB_n = (-1)^n a_n I$$

On multiplying the equation by $A^n, A^{n-1}.....I$ respectively and adding, we obtain

$$0 = (-1)^n \left[A^n + a_1 A^{n-1} ..... a_n I \right]$$
$$= A^n + a_1 A^{n-1} ..... a_n I = 0$$

Thus if A is square matrix and if

$$\lambda^3 - 2\lambda^2 + 3\lambda - 4 = 0 \qquad \text{..... (1)}$$

be its characteristic equation, then by Caley-Hamilton theorem (1) is satisfied by A.

$$A^3 - 2A^2 + 3A - 4I = 0 \qquad \text{..... (2)}$$

We can find out A^{-1} from (2)

On pre multiplying (2) by A^{-1}, we get

$$A^2 - 2A + 3I - 4A^{-1} = 0$$
$$A^{-1} = \frac{1}{4}\left[A^2 - 2A + 3I \right]$$

Problem: Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

verify Caley-Hamilton theorem and hence evaluate the matrix equation $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

Solution: Characteristic equation of matrix A is

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix} = 0 \ \ or \ \ \left[(2-\lambda)(1-\lambda)(2-\lambda)\right] - 1(0) + 1(0 - 1 + \lambda) = 0$$

or $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

According to Caley Hamilton Theorem

$$A^3 - 5A^2 + 7A - 3I = 0$$

Verification $A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$$A^3-5A^2+7A-3I=\begin{bmatrix}14 & 13 & 13\\ 0 & 1 & 0\\ 13 & 13 & 14\end{bmatrix}-5\begin{bmatrix}5 & 4 & 4\\ 0 & 1 & 0\\ 4 & 4 & 5\end{bmatrix}+7\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}\text{-}3\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=0$$

Caley-Hamilton Theorem is verified.

Now $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$

$$A^5\left[A^3-5A^2+7A-3I\right]+A\left[A^3-5A^2+7A-3I\right]+A^2+A+I$$

$$A^5\times 0+A\times 0+A^2+A+I=A^2+A+I$$

$$=\begin{bmatrix}5 & 4 & 4\\ 0 & 1 & 0\\ 4 & 4 & 5\end{bmatrix}+\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}+\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}8 & 5 & 5\\ 0 & 3 & 0\\ 5 & 5 & 8\end{bmatrix}$$

Lecture 7: Characteristic Vectors or Eigen Vectors

Corresponding to each characteristic root λ, we have a corresponding non zero vector X which satisfies the equation $\left|A-\lambda I\right|X=0$ and this non-zero vector X is called characteristic vector or Eigen Vectors.

Properties of Eigen vectors:

1. Eigen Vector X of a matrix is not unique.
2. If $\lambda_1,\lambda_2.....\lambda_n$ be distinct eigenvalues of an $n\times n$ matrix then corresponding Eigen vectors $X_1,X_2.....X_n$ form a linearly independent set.
3. If two or more Eigen values are equal, then it may or may not be possible to get linearly independent Eigen vectors corresponding to the equal roots.
4. Two Eigen vectors X_1 and X_2 are called orthogonal vectors if $X_iX_2 = 0$
5. Eigen vectors of a symmetric matrix Corresponding to different Eigen values are orthogonal.

Normalized form of vector's : To find normalized form of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

we divide each element by $\sqrt{a^2+b^2+c^2}$

For example, Normalized form of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} is \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

Non symmetric matrices with non-repeated Eigen values

Problem : Find the Eigen values and Eigen vectors of the matrix $A=\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Solution : $|A-\lambda I|=0$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}=0 \ \ i.e. \ \ \lambda^3-6\lambda^2+11\lambda-6=0$$

Let $\lambda=1,$ then $\lambda^3-6\lambda^2+11\lambda-6=0$

So $(\lambda-1)(\lambda^2-5\lambda+6)=0$

$(\lambda-1)(\lambda-2)(\lambda-3)=0 \Rightarrow \lambda=1,2,3$

To find Eigen vectors for the corresponding Eigen values we consider the matrix equation $(A-\lambda I)X=0$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigen Vectors corresponding to $\lambda=1$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\Rightarrow\begin{bmatrix} -z=0 \\ x+y+z=0 \\ x+y+z=0 \end{bmatrix}$$

So $z=0$

& if x = k, then y = -k

Hence Eigen Vector X_1 is $\begin{bmatrix} k \\ -k \\ 0 \end{bmatrix}$ *or* $k\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Eigen vectors corresponding to $\lambda = 2$ is

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -x+0y-z=0 \\ 2x+2y+z=0 \\ \end{bmatrix}$$

$$\frac{x}{0+2} = \frac{y}{-2+1} = \frac{z}{-2-0} \quad or \quad \frac{x}{2} = \frac{y}{-1} = \frac{z}{-2}$$

Hence Eigen vector $X_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

Vector corresponding to $\lambda = 3$

$$\begin{bmatrix} 0 & -1 & -1 \\ -1 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2x+0y-z=0 \\ x-y+z=0 \\ \end{bmatrix}$$

$$\frac{x}{0-1} = \frac{y}{-1+z} = \frac{z}{2-0} \quad or \quad \frac{x}{1} = \frac{y}{-1} = \frac{z}{-2}$$

$$X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

Non Symmetric matrix with repeated Eigen Values:

Problem: Find all the Eigen Values and Eigen Vectors

of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution: Characteristic Equation of A is

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} = 0$$

or $(-2-\lambda)\left[-\lambda^2+\lambda-12\right]-2(-2\lambda-6)-3(-4+1-\lambda)=0$

or $\lambda^3+\lambda^2-21\lambda-45=0$

If $\lambda = -3,$ *then* $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

so $\lambda^2 - 2\lambda - 15 = 0$ *or* $(\lambda - 5)(\lambda + 3) = 0$

Eigen values are 5, -3, -3

To find Eigen vectors for these corresponding Eigen values, we have

$$(A - \lambda I)X = 0 \text{ i.e. } \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = 5$, we get

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have $\frac{x}{-12-12} = \frac{y}{-6-42} = \frac{z}{28-4}$ *or* $\frac{x}{24} = \frac{y}{-48} = \frac{z}{-24}$ *or* $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1} = k$

Hence Eigen Vector is $X_1 = \begin{bmatrix} k \\ 2k \\ -k \end{bmatrix}$ *or* $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

For $\lambda = -3, \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We have $x + 2y - 3z = 0,\ \ 2x + 4y - 6z = 0,\ \ -x - 2y + 3z = 0$

Some equations, so let $x = k_1,\ \ y = k_2,\ \ \text{then } z = \frac{1}{3}(k_1 + 2k_1)$

Hence the Eigen vector is $X_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

Since the matrix in non-symmetric the corresponding Eigen vectors X_2 and X_3 must be linearly independent which is possible by choosing $k_2 = 0$, $k = 1$

Hence $X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

Lecture 8: Diagonalization of a Matrix:

If a square matrix A of order n has n linearly independent Eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix So $D = PAP^{-1}$

Remark:

1- The square matrix P, which diagonalises A. is found by grouping the Eigen vectors of A into square matrix and the resulting diagonal matrix has the Eigen values of A as its diagonal elements.

2- The transformation of a matrix A to $P^{-1}AP$ is known as a similarity transformation.

3- The matrix P which diagonalises A is called the model matrix of A and the resulting diagonal matrix D is known as the spectra matrix of A.

Problem : A square matrix A is defined by $A=\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$,

find the model matrix P and the resulting diagonal matrix D of A.

Solution: The characteristic equation of the matrix A is

$$\begin{bmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{bmatrix}=0$$

or $(1-\lambda)(-2\lambda+\lambda^2+1)-2(-\lambda+1)-2(-1+2-\lambda)=0$

or $(-\lambda+1)(\lambda^2-2\lambda+1)+2\lambda-2-2+2\lambda=0$

or $(-\lambda+1)(\lambda-1)^2+(4\lambda-4)=0 \Rightarrow (1-\lambda)\left[(\lambda-1)^2-4\right]=0$

or $\Rightarrow (1-\lambda)(\lambda^2-2\lambda-3)=0 \Rightarrow (1-\lambda)(\lambda+1)(\lambda-3)=0$

Eigen values are 1, -1, 3

Eigen vectors are given by the matrix equation.

$$\begin{bmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 1$, $\begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We have 0.x + 2y - 2z = 0

x + y + z = 0

From (1) & (2) equation, we have

$$\frac{x}{2+2} = \frac{y}{-2-0} = \frac{z}{0-2} \quad or \quad \frac{x}{4} = \frac{y}{-2} = \frac{z}{-2} \Rightarrow x = -2, y = 1, z = 1$$

So $X_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

When $\lambda = -1$, $\begin{bmatrix} 1+1 & 2 & -2 \\ 1 & 2+1 & 1 \\ -1 & -1 & 0+1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or $\begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

we have $2x + 2y - 2z = 0, \quad x + 3y + z = 0, \quad -x - y + z = 0$

From (2) & (3) equation we have

$$\frac{x}{3+1} = \frac{y}{-1-1} = \frac{z}{-1+3} \quad or \quad \frac{x}{4} = \frac{y}{-2} = \frac{z}{2} \quad or\ x = -2, y = 1, z = -1$$

So $X_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

When $\lambda = 3$, $\begin{bmatrix} 1-3 & 2 & -2 \\ 1 & 2-3 & 1 \\ -1 & -1 & 0-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} -2 & 2 & -2 \\ 1 & -1 & 1 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We have $-2x + 2y - 2z = 0, \quad x - y + z = 0, \quad -x - y - 3z = 0$

From (2) & (3) equation we have

$$\frac{x}{3+1} = \frac{y}{-1+3} = \frac{z}{-1-1} \quad or \quad \frac{x}{4} = \frac{y}{2} = \frac{z}{-2} \quad or\ x = 2, y = 1, z = -1$$

Hence $X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

Now Model matrix $P = \begin{bmatrix} -2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

Matrix of cofactors $= \begin{bmatrix} 0 & 2 & -2 \\ -4 & 0 & -4 \\ -4 & 4 & 0 \end{bmatrix}$

$$P^{-1} = -\frac{1}{8}\begin{bmatrix} 0 & 2 & -2 \\ -4 & 0 & -4 \\ -4 & 4 & 0 \end{bmatrix}$$

For diagonalization $D = P^{-1}AP = -\frac{1}{8}\begin{bmatrix} 0 & -4 & -4 \\ 2 & 0 & 4 \\ -2 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -1 & -1 & -1 \end{bmatrix}$

$$\begin{bmatrix} -2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$= -\frac{1}{8}\begin{bmatrix} -8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -24 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

So D = diag (1, -1, 3)

Remark:

Power of a matrix $D = P^{-1}AP$

$$D^2 = P^{-1}A^2P$$

.....

$$D^n = P^{-1}A^nP$$

$$PD^nP^{-1} = P\left(P^{-1}A^nP\right)P^{-1} = \left(PP^{-1}\right)D^n\left(PP^{-1}\right) = A^n$$

Problem: Find a matrix P which transform the matrix A

$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form. Hence find A^4

Solution: Characteristic equation of the matrix A is

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}=0 \text{ or } \lambda^3-6\lambda^2+11\lambda-6=0 \text{ or } (\lambda-1)(\lambda-2)(\lambda-3)=0$$

For $\lambda=1$, Eigen vectors are $x_1=1, x_2=-1, x_3=0$

For $\lambda=2$, Eigen vectors are $x_1=-2, x_2=1, x_3=3$

For $\lambda=3$, Eigen vectors are $x_1=-1, x_2=1, x_3=2$

Hence Model Matrix $P=\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & -1 \\ 0 & 2 & 2 \end{bmatrix}$ & $P^{-1}=-\frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$

$$\& \quad P^{-1}AP=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}=D$$

$$A^4=PD^4P^{-1}=\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix}$$

$$=\begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

Unit-I

Tutorial-01 (Matrices), Semester-II KAS 203 MATHEMATICS-II

Prepared by Prof. Raj Kumar

Section A (Rank)

1. Each entry of a matrix is unity. Show that its rank is one.
2. If A is a non-zero column vector $(n \times 1)$ then rank of AA^T *is* (A)2, (B)n, (C)0, (d*)1 .
3. If $\rho(A_{3\times3})=2$ and $\rho(B_{3\times3})=3$ then $\rho(AB)$= (a) 5, (b)3,(c*)2 ,(d) 1
4. If A_{5x7} has all entries -1 then rank of A is (a) 7; (b*) 1; (c) 5; (d) 0.
5. If rank of $\begin{bmatrix} K & -1 & 0 \\ 0 & K & -1 \\ -1 & 0 & K \end{bmatrix}$ is 2, then K is (a) Arbitrary (b*) 1, (c) 2, (d)3.
6. If A= $\begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 2 & 3 & 10 \end{bmatrix}$, B= $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$, Find $\rho(A)$, $\rho(B)$, $\rho(A+B)$, $\rho(AB)$ and $\rho(BA)$.

   Ans: $\rho(A)=2$, $\rho(B)=1$, $\rho(AB)=1$and , $\rho(BA)=1$ Hint: $\rho\ (Product) \leq \rho(Either)$

7. Determine the value of b if the rank of A *is* 3, where $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{bmatrix}$

   {Hint: After using Row transformations

   $R_4 \leftrightarrow R_3$, $\sim$, $R_4 - 3R_2$ $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & b+6 & 0 \\ b-2 & 0 & 0 & -2 \end{bmatrix}$ Cases: (1) If b=2, Then $|A| = 0$, $\rho(A) = 3$.

   (2) If $b = -6$ Then, number of non-zero rows is 3. Therefore $\rho(A) = 3$.

8. Reducing A= $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$ to its Normal form, find the rank of the matrix? Ans: 3.
9. Find rank of (i) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 5 & 2 \end{bmatrix}$ Ans: 3. (ii) A=$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ Ans: 2 (iii) B= $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ Ans: 2

   (iv) A=$\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -3/2 \end{bmatrix}$ **Ans:** $\rho(A)=1$ **(v)** $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ **Ans: 2**

   (vi) $\begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & -6 \end{bmatrix}$ **Ans:** $\rho = 4$. (vii) $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$ ***Ans***. 3

   (viii) $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & 15 \end{bmatrix}$ ***Ans***. 2 (ix) $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ ***Ans***. 3

Section B (Symmetric, Skew-symmetric, Orthogonal, Hermitian and Skew Hermitian Matrices)

1. Define symmetric, skew-symmetric, and orthogonal matrices with one example.
2. Let A be a square matrix over C. Show that $A+A^{\theta}$, AA^{θ}, and $A^{\theta}A$ are Hermitian matrices and $A-A^{\theta}$ a skew-Hermitian matrix.

3. If A=$\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 4\\ 1 & 4 & 3\end{bmatrix}$ then |AdjA| is (a) -4, (b*)4, (c)-2 ,(d) 2
4. If A and B are square matrices of the same order, which of the following is true
(a)$(A+B)^2 = A^2 + 2AB + B^2$,$(b)(A+B)(A-B) = A^2 - B^2$,
$(c)\ (A-B)\ (A+B)\ = A^2 - B^2$,$(d*)\ (A+B)(A-B) + \ (A-B)\ (A+B)\ = 2A^2 - 2B^2$
5. Show that for any square matrix A with real number entries, A + A′ is a symmetric matrix and A – A′ is a skew-symmetric matrix.
6. If A be a square matrix such that A,A^2, $A^3$are non-zero matrices but A^4 is a zero matrix.Then $(I - A)^{-1}$ is (a) A+ $A^2 + A^3$(b)I+A+ A^2 (c) I+$A^2 + A^3$, (D*) I+A+ $A^2 + A^3$
7. If A=$\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$, show that $A^n = \begin{bmatrix}\cos n\alpha & \sin n\alpha\\ -\sin n\alpha & \cos n\alpha\end{bmatrix}$, where n∈$\mathbb{Z}^+$
8. Prove (i) $\begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix}$ is a nilpotent matrix of index 3; (ii). $\begin{bmatrix}1 & 0 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & -2 & 1 & 0\\ 1 & -3 & 3 & -1\end{bmatrix}$ is an involutory matrix (iii) $\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$ is an orthogonal matrix.; (iv) $\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}$ is a unitary matrix.
9. Express $\begin{bmatrix}3 & 3 & -1\\ -2 & -2 & 1\\ -4 & -5 & 2\end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.
10. Check A=$\begin{bmatrix}1 & -1 & 5\\ -1 & 2 & 1\\ 5 & 1 & 3\end{bmatrix}$ B=$\begin{bmatrix}1 & i & 3i\\ -i & 0 & 2-i\\ -3i & 2+i & -1\end{bmatrix}$ and C=$\begin{bmatrix}1 & i & 3i\\ -i & 0 & 2-i\\ -3i & 2+i & -1\end{bmatrix}$ for symmetric matrix, Hermitian.
11. If k is complex number and A be a Hermitian matrix. Will kA be Hermitian?
12. If matrix A=$\begin{bmatrix}i & 1+i & 2-3i\\ -1+i & 2i & 1\\ -2-3i & -1 & 0\end{bmatrix}$ then prove that the conjugate of A is skew Hermitian
13. Check whether matrix $\begin{bmatrix}3 & 1+i & 2-3i\\ -1+i & 2i & 1\\ -2-3i & -1 & 0\end{bmatrix}$ is a skew-Hermitian matrix.

Section C (Characteristic equation, Cayley-Hamilton theorem, Diagonalization of a Matrix)

1. Prove that all the eigen values of a Hermitian matrix are real.
2. Prove that the eigen values of a triangular matrix are just diagonal elements of the matrix.
3. Find all the eigen values and eigenvectors of the matrices
(i) A=$\begin{bmatrix}3 & 1 & 4\\ 0 & 2 & 6\\ 0 & 0 & 5\end{bmatrix}$Ans: 2,3,5 ;$\begin{bmatrix}1\\ -1\\ 0\end{bmatrix}$, $\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$, $\begin{bmatrix}3\\ 2\\ 1\end{bmatrix}$ (ii)B=$\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ Ans: -3,-3,5 ;$\begin{bmatrix}c\\ d\\ \frac{c+2d}{3}\end{bmatrix}$, $\begin{bmatrix}-c\\ -2c\\ c\end{bmatrix}$,
4. Verify Cayley-Hamilton theorem for the following matrices and hence find A^{-1}
(i) A=$\begin{bmatrix}4 & 3 & 1\\ 2 & 1 & -2\\ 1 & 2 & 1\end{bmatrix}$ $A^{-1}=\frac{1}{11}\begin{bmatrix}5 & -1 & -7\\ -4 & 3 & 10\\ 3 & -5 & -2\end{bmatrix}$, (ii)A=$\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ $A^{-1}=\frac{1}{4}\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$

5. A square matrix is defined by A=$\begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$, Is it diagonalizable? If yes, reduce it into diagonal form. Ans: Diag$(1, -1, 3)$
6. Prove that if λ is an eigen value of a matrix A, then kλ, $\boldsymbol{\lambda \pm k}$, $\lambda^n, \frac{1}{\lambda}, \frac{|A|}{\lambda}$ are the eigen values of kA, A$\pm kI, A^n, A^{-1}$, Adj A respectively.

Section D (System of linear equations)

1. If the system of equations $3x - y + \lambda z = 1, 2x + y + z = 2, x + 2y - \lambda z = -1$ has a unique solution if λ has (a) any value, (b) $\frac{-7}{2}$, (c*) $\neq \frac{-7}{2}$, (d) $\neq \frac{7}{2}$.
2. Check the consistency for the following linear systems. If system is consistence find the

$x_1 + x_2 + 2x_3 + x_4 = 5$ $2x_1 + 3x_2 - x_3 - 2x_4 = 2$ $4x_1 + 5x_2 + 3x_3 = 7$, **Ans: No solution**	$2x + 6y + 11 = 0$ $6x + 20y - 6z = -3$ $6y - 18z + 1 = 0$ **Ans: No solution**	$x + 2y - z = 3$ $3x - y + 2z = 1$ $2x - 2y + 3z = 2$ $x - y + z = -1$ **Ans: $x = -1, y = 4, z = 4$**

3. If the system of equations $x + 4ay + az = 0, x + 3by + bz = 0, x + 2cy + cz = 0$ has a non-trivial solution, then a, b, c, are in (a) A.P, (b)G.P, (c*) H.P., (d) None of these.
4. For what values of λ and μ so that the equations $x + y + z = 6; x + 2y + 3z = 10; x + 2y + \lambda z = \mu$ have (i) No solution; (ii) a unique solution and (iii) an infinite number of solutions ANS**:** (i) $\lambda = 3$ no solution; (ii)$\lambda \neq 3$, $\mu \neq 10$, unique solution; (iii)$\lambda = 3$, $\mu = 10$, Infinite many solutions.
5. Determine b such that the system $2x + y + 2z = 0; x + y + 3z = 0; 4x + 3y + bz = 0$ has (i) trivial solution(ii) non- trivial solution**.**
6. Find the values of a , b and c for which the system has (i) No solution (ii) unique solution (iii) Infinitely many solutions for

$-2x + y + z = a;$ $x - 2y + z = b;$ $x + y - 2z = c$ **Ans**: $(i) If\ a + b + c \neq 0$ *No solution* $(ii) If\ a + b + c = 0$,Infinite solutions	$x + ay + z = 3$ $x + 2y + 2z = b$ $x + 5y + 3z = 9$ **Ans:** (i) $a = -1, b \neq 6$ *no solution* (ii)$a \neq -1, b$ *is arbitrary, unique solution* (iii)$a = -1, b = 6$ *Infinite many* solutions.